ELEN E4810: Digital Signal Processing

Topic 8:
Filter Design: IIR

1. Filter Design Specifications
2. Analog Filter Design
3. Digital Filters from Analog Prototypes

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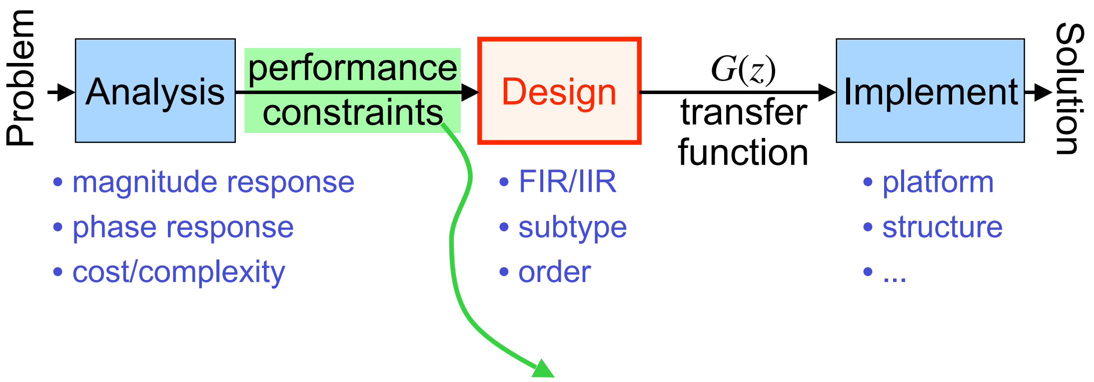
1. Filter Design Specifications

- The filter design process:



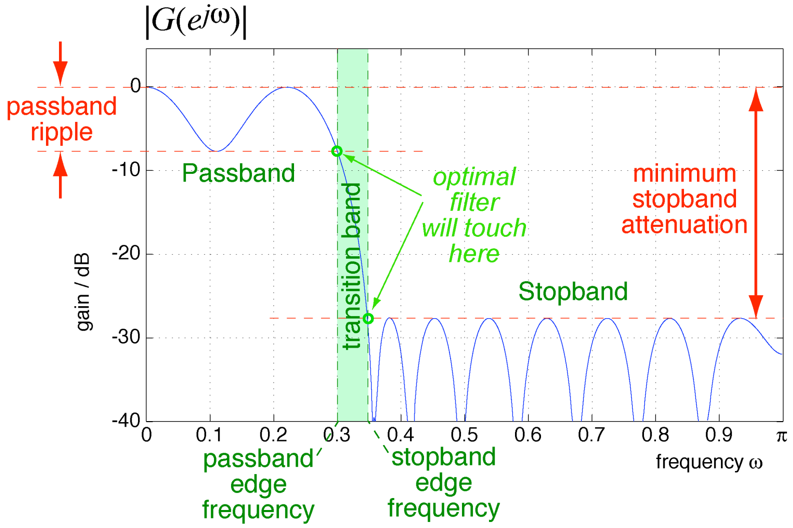
- magnitude response
- phase response
- cost/complexity

- FIR/IIR
- subtype
- order

- platform
- structure
- ...

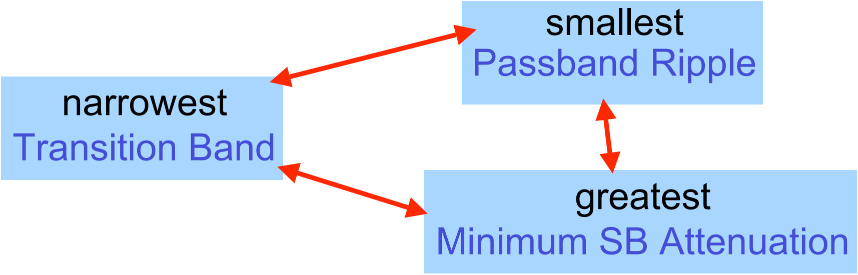
Performance Constraints

- .. in terms of magnitude response:



Performance Constraints

- “Best” filter:

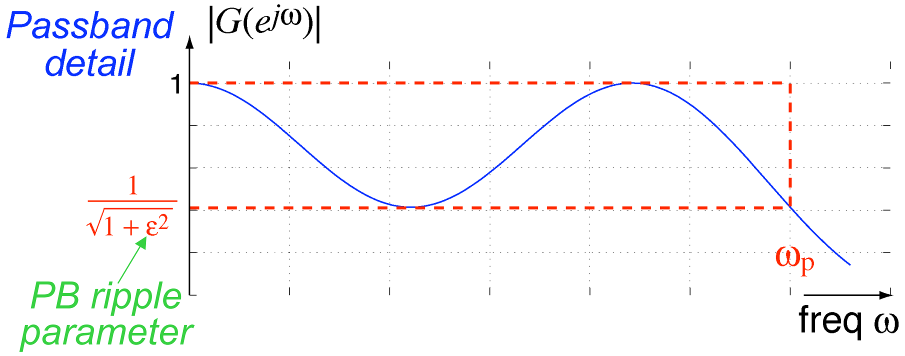

narrowest Transition Band

smallest Passband Ripple

greatest Minimum SB Attenuation

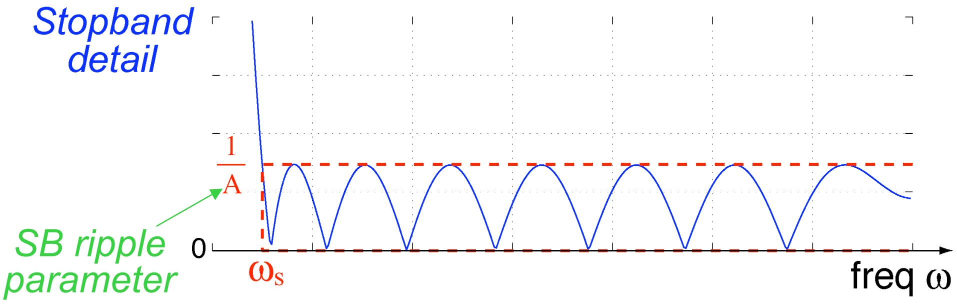
  - improving one usually worsens others
- But: increasing filter order (i.e. cost) improves all three measures

Passband Ripple



- Assume peak passband gain = 1 then *minimum* passband gain = $\frac{1}{\sqrt{1+\varepsilon^2}}$
- Or, ripple $\alpha_{\max} = 20\log_{10}\sqrt{1+\varepsilon^2}$ dB

Stopband Ripple



- Peak passband gain is $A\times$ larger than peak stopband gain
- Hence, minimum stopband attenuation $\alpha_s = -20\log_{10}\frac{1}{A} = 20\log_{10}A$ dB

Filter Type Choice: FIR vs. IIR

	FIR	**IIR**
	No feedback (just zeros)	Feedback (poles & zeros)
	Always stable	May be unstable
	Can be linear phase	Difficult to control phase
BUT	High order (20-2000)	Typ. < 1/10th order of FIR (4-20)
	Unrelated to continuous-time filtering	Derive from *analog prototype*

FIR vs. IIR

- If you care about computational cost
  → use low-complexity IIR
  (computation no object → Lin Phs FIR)
- If you care about phase response
  → use linear-phase FIR
  (phase unimportant → go with simple IIR)

IIR Filter Design

- IIR filters are directly related to analog filters (continuous time)
  - via a mapping of $H(s)$ (CT) to $H(z)$ (DT) that preserves many properties
- Analog filter design is sophisticated
  - signal processing research since 1940s

→ Design IIR filters via *analog prototype*

  - hence, need to learn some CT filter design

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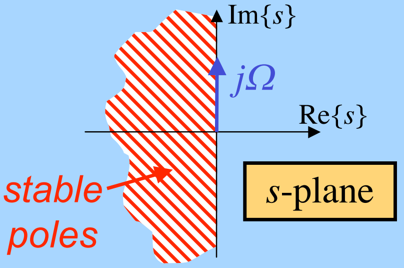
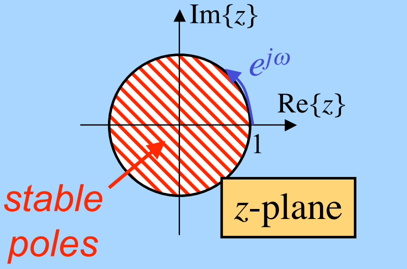
2. Analog Filter Design

- Decades of analysis of transistor-based filters – sophisticated, well understood
- Basic choices:
  - ripples vs. flatness in stop and/or passband
  - more ripples → narrower transition band

Family	*PB*	*SB*
Butterworth	flat	flat
Chebyshev I	ripples	flat
Chebyshev II	flat	ripples
Elliptical	ripples	ripples

CT Transfer Functions

- Analog systems: s-transform (Laplace)

	Continuous-time	*Discrete-time*
Transform	$H_a(s) = \int h_a(t) e^{-st} dt$	$H_d(z) = \sum h_d[n] z^{-n}$
Frequency response	$H_a(j\Omega)$	$H_d\left(e^{j\omega}\right)$
Pole/zero diagram		

Butterworth Filters

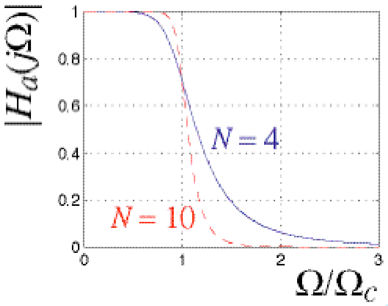
Maximally flat in pass and stop bands

- Magnitude response (LP): $$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

filter order N

  - $\Omega \ll \Omega_c$, $|H_a(j\Omega)|^2 \rightarrow 1$
  - $\Omega = \Omega_c$, $|H_a(j\Omega)|^2 = {}^1/_2$

3dB point

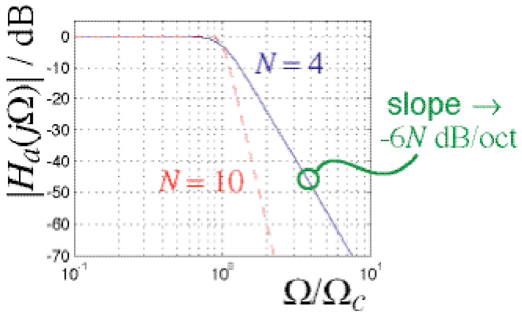


Butterworth Filters

6N dB/oct rolloff

- $\Omega >> \Omega_c, \quad |H_a(j\Omega)|^2 \rightarrow (\Omega_c/\Omega)^{2N}$

Log-log magnitude response

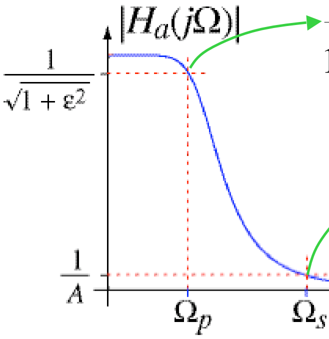


- *flat* $\rightarrow \dfrac{d^n}{d\Omega^n}\left|H_a(j\Omega)\right|^2 = 0$

@ Ω = 0 for $n = 1 .. 2N$-1

Butterworth Filters

- How to meet design specifications?



$$\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \geq \frac{1}{1+\varepsilon^2}$$

$$\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \leq \frac{1}{A^2}$$

Design Equation

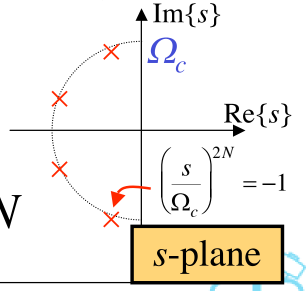
$$N \geq \frac{1}{2}\frac{\log_{10}\left(\frac{A^2-1}{\varepsilon^2}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

- $k_1 = \dfrac{\varepsilon}{\sqrt{A^2-1}}$ ="discrimination", <<1
- $k = \dfrac{\Omega_p}{\Omega_s}$ ="selectivity", < 1

Butterworth Filters

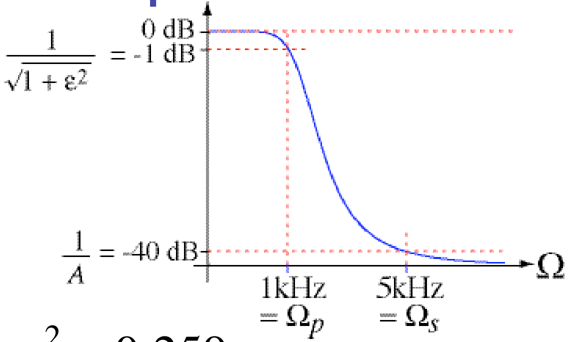
- $|H_a(j\Omega)|^2 = \frac{1}{1+(\frac{\Omega}{\Omega_c})^{2N}}$ but what is $H_a(s)$?
- Traditionally, look it up in a table
  - calculate $N \rightarrow$ normalized filter with $\Omega_c = 1$
  - scale all coefficients for desired Ω_c
- In fact, $H_a(s) = \frac{1}{\prod_i (s - p_i)}$

  where $p_i = \Omega_c e^{j\pi\frac{N+2i-1}{2N}} \quad i = 1..N$



Butterworth Example

Design a Butterworth filter with 1 dB cutoff at 1kHz and a minimum attenuation of 40 dB at 5 kHz



$$-1\text{dB} = 20\log_{10}\frac{1}{\sqrt{1+\varepsilon^2}} \Rightarrow \varepsilon^2 = 0.259$$

$$-40\text{dB} = 20\log_{10}\tfrac{1}{A} \Rightarrow A = 100$$

$$\Omega_s / \Omega_p = 5$$

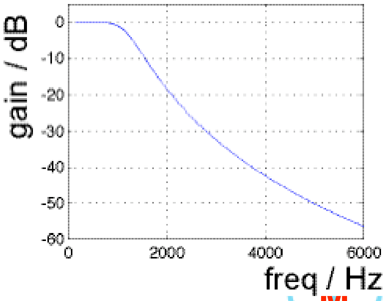
$$N \geq \frac{1}{2}\frac{\log_{10}\frac{9999}{0.259}}{\log_{10} 5}$$

$$\Rightarrow N = 4 \geq 3.28$$

Butterworth Example

- Order $N = 4$ will satisfy constraints; What are Ω_c and filter coefficients?
  - from a table, $\Omega_{\text{-1dB}} = 0.845$ when $\Omega_c = 1$ $\Rightarrow \Omega_c = 1000/0.845 = 1.184\ \text{kHz}$
  - from a table, get normalized coefficients for $N = 4$, scale by $1184 \cdot 2\pi$
- Or, use Matlab:

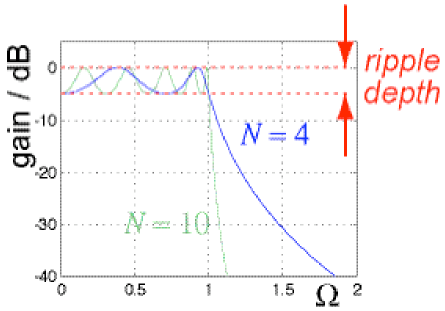
```
[b,a] =
butter(N,Wc,’s’);
```



Chebyshev I Filter

- Equiripple in passband (flat in stopband) → minimize maximum error

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\frac{\Omega}{\Omega_p})}$$



Chebyshev polynomial of order N →

$$T_N(\Omega) = \begin{cases} \cos\left(N \cos^{-1}\Omega\right) & |\Omega| \le 1 \\ \cosh\left(N \cosh^{-1}\Omega\right) & |\Omega| > 1 \end{cases}$$

Chebyshev I Filter

- Design procedure:
  - desired passband ripple → ε
  - min. stopband atten., Ω_p, Ω_s → N:

$$\frac{1}{A^2} = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{1}{1+\varepsilon^2\left[\cosh\left(N\cosh^{-1}\frac{\Omega_s}{\Omega_p}\right)\right]^2}$$

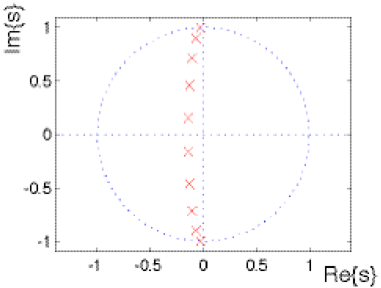
$$\Rightarrow N \geq \frac{\cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$1/k_1$, *discrimination*

$1/k$, *selectivity*

Chebyshev I Filter

- What is $H_a(s)$?
  - complicated, get from a table
  - .. or from Matlab `cheby1(N,r,Wp,'s')`
  - all-pole; can inspect them:



..like squashed-in Butterworth

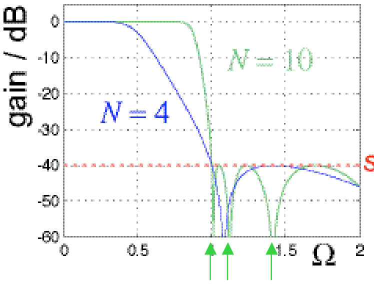
Chebyshev II Filter

- Flat in passband, equiripple in stopband

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\dfrac{T_N(\frac{\Omega_s}{\Omega_p})}{T_N(\frac{\Omega_s}{\Omega})}\right)^2}$$

constant

$\sim 1/T_N(1/\Omega)$



peak of stopband ripples

zeros on imaginary axis

- Filter has poles and zeros (some)
- Complicated pole/zero pattern

Elliptical (Cauer) Filters

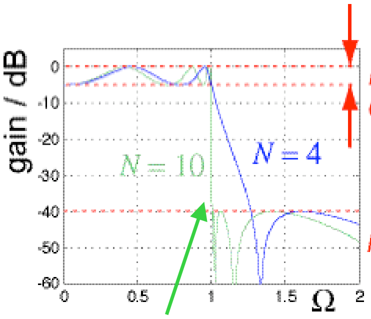
- Ripples in both passband and stopband

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 R_N^2(\frac{\Omega}{\Omega_p})}$$

function; satisfies

$R_N(\Omega^{-1}) = R_N(\Omega)^{-1}$

zeros for $\Omega<1$ → poles for $\Omega>1$



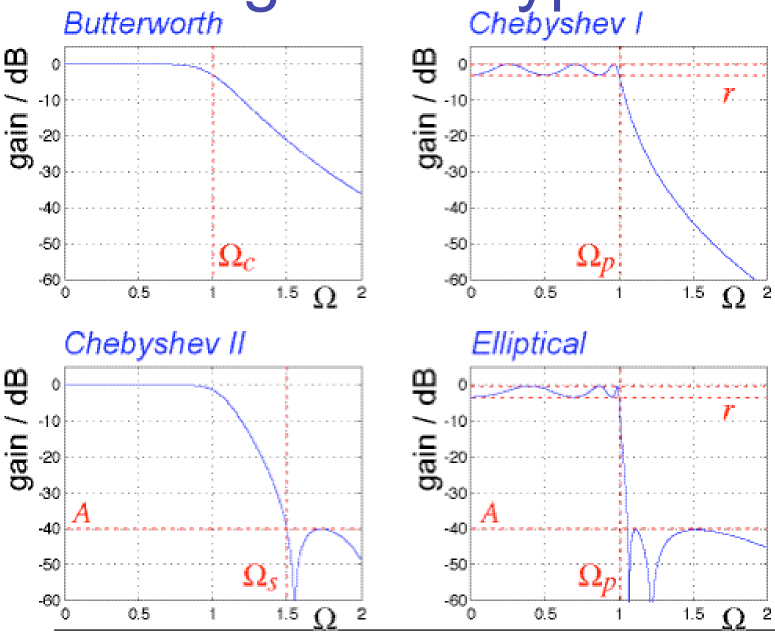
PB ripple depth

peak SB ripples

very narrow transition band

- Complicated; not even closed form for N

Analog Filter Types Summary



Butterworth

Chebyshev I

Chebyshev II

Elliptical

$N = 6$

$r = 3$ dB

$A = 40$ dB

Analog Filter Transformations

- All filters types shown as lowpass; other types (highpass, bandpass..) derived via transformations
- i.e. $\hat{s} = F^{-1}(s)$

lowpass prototype $H_{LP}(s) \quad \rightarrow \quad H_D(\hat{s})$ Desired alternate response; still a ***rational polynomial***

- General mapping of s-plane *BUT* keep $j\Omega \rightarrow j\hat{\Omega}$; frequency response just ‘shuffled’

Lowpass-to-Highpass

- Example transformation:

$$H_{HP}(\hat{s}) = H_{LP}(s)\Big|_{s=\frac{\Omega_p\hat{\Omega}_p}{\hat{s}}}$$

  - take prototype $H_{LP}(s)$ polynomial
  - replace s with $\frac{\Omega_p\hat{\Omega}_p}{\hat{s}}$
  - simplify and rearrange
    $\rightarrow$ new polynomial $H_{HP}(\hat{s})$

Lowpass-to-Highpass

- What happens to frequency response?

$$s = j\Omega \quad \Rightarrow \quad \hat{s} = \frac{\Omega_p\hat{\Omega}_p}{j\Omega} = j\left(\frac{-\Omega_p\hat{\Omega}_p}{\Omega}\right)$$

imaginary axis stays on self...

$$\Rightarrow \hat{\Omega} = \frac{-\Omega_p\hat{\Omega}_p}{\Omega}$$

...freq.→freq.

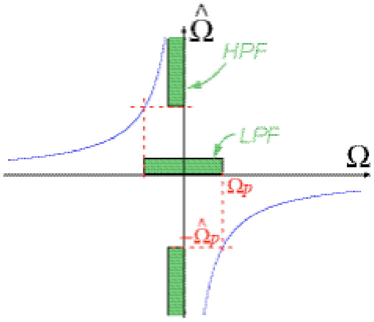
- $\Omega = \Omega_p \quad \rightarrow \quad \hat{\Omega} = -\hat{\Omega}_p$

  $\Omega < \Omega_p \quad \rightarrow \quad \hat{\Omega} < -\hat{\Omega}_p$

  LP passband *HP passband*

  $\Omega > \Omega_p \quad \rightarrow \quad \hat{\Omega} > -\hat{\Omega}_p$

  LP stopband *HP stopband*



- Frequency axes inverted

Transformation Example

Design a Butterworth highpass filter with PB edge -0.1dB @ 4 kHz ($\hat{\Omega}_p$) and SB edge -40 dB @ 1 kHz ($\hat{\Omega}_s$)

- Lowpass prototype: make $\Omega_p = 1$

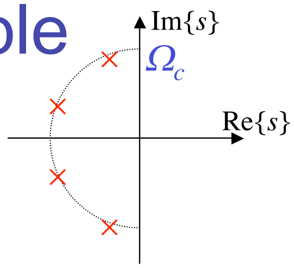
$$\Rightarrow \Omega_s = (-)\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}_s} = (-)4$$

- Butterworth -0.1dB @ Ω_p=1, -40dB @ Ω_s=4

$$N \geq \frac{1}{2}\frac{\log_{10}\left(\frac{A^2-1}{\varepsilon^2}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)} \qquad \Omega_p \;@ -0.1dB \Rightarrow \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{10}} = 10^{\frac{-0.1}{10}}$$

$$\rightarrow N = 5 \qquad \rightarrow \Omega_c = \Omega_p / 0.6866 = 1.4564$$

Transformation Example



- LPF proto has $p_\ell = \Omega_c e^{j\pi\frac{N+2\ell-1}{2N}}$

$$\Rightarrow H_{LP}(s) = \frac{\Omega_c^N}{\prod_{\ell=1}^{N}(s-p_\ell)}$$

- Map to HPF: $H_{HP}(\hat{s}) = H_{LP}(s)\big|_{s=\frac{\Omega_p\hat{\Omega}_p}{\hat{s}}}$

$$\Rightarrow H_{HP}(\hat{s}) = \frac{\Omega_c^N}{\prod_{\ell=1}^{N}\left(\frac{\Omega_p\hat{\Omega}_p}{\hat{s}} - p_\ell\right)} = \frac{\Omega_c^N \hat{s}^N}{\prod_{\ell=1}^{N}\left(\Omega_p\hat{\Omega}_p - p_\ell\hat{s}\right)}$$

N zeros @ $\hat{s} = 0$

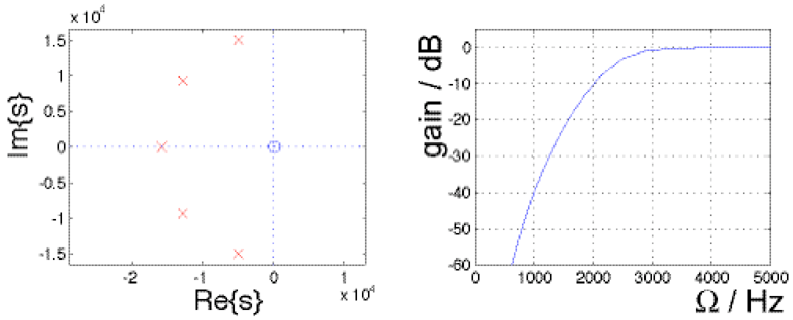
new poles @ $\hat{s} = \Omega_p\hat{\Omega}_p / p_l$

Transformation Example

- In Matlab:

Ω_p Ω_s R_p R_s

```
[N,Wc]=buttord(1,4,0.1,40,'s');
[B,A] = butter(N, Wc, 's');
[n,d] = lp2hp(B,A,2*pi*4000);
```



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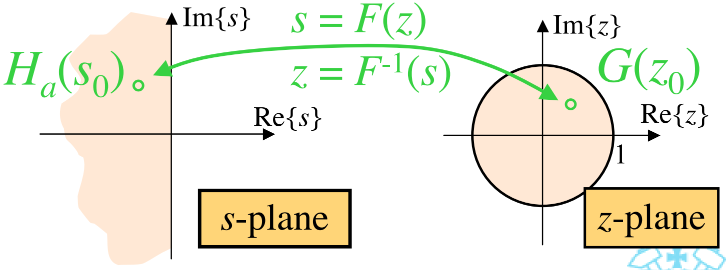
3. Analog Protos → IIR Filters

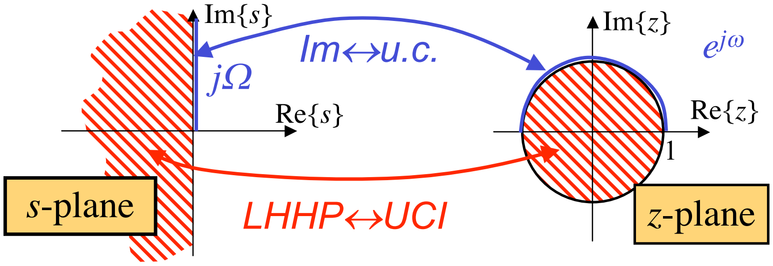
- Can we map high-performance CT filters to DT domain?
- Approach: **transformation** $H_a(s) \rightarrow G(z)$
  i.e. $G(z) = H_a(s)\big|_{s=F(z)}$
  where $s = F(z)$ maps s-plane ↔ z-plane:



Every value of $G(z)$ is a value of $H_a(s)$ somewhere on the s-plane & vice-versa

CT to DT Transformation

- Desired properties for $s = F(z)$:
  - s-plane $j\Omega$ axis ↔ z-plane unit circle
    → preserves frequency response values
  - s-plane LHHP ↔ z-plane unit circle interior
    → preserves stability of poles



Bilinear Transformation

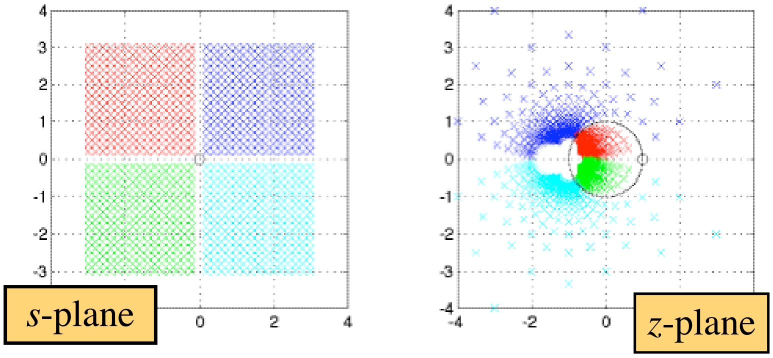
- Solution: $$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$ **Bilinear Transform**
- Hence inverse: $z = \dfrac{1+s}{1-s}$ *unique, 1:1 mapping*
- Freq. axis? $s = j\Omega \quad \rightarrow \quad z = \dfrac{1+j\Omega}{1-j\Omega}$ $|z| = 1$ *i.e. on unit circle* ✓
- Poles? $s = \sigma + j\Omega \quad \rightarrow \quad z = \dfrac{(1+\sigma)+j\Omega}{(1-\sigma)-j\Omega}$

$$\Rightarrow |z|^2 = \frac{1+2\sigma+\sigma^2+\Omega^2}{1-2\sigma+\sigma^2+\Omega^2}$$

$\sigma < 0$
$\leftrightarrow |z| < 1$ ✓

Bilinear Transformation

- How can entire half-plane fit inside u.c.?



- Highly nonuniform warping!

Bilinear Transformation

- What is CT↔DT freq. relation Ω↔ω ?

$$z = e^{j\omega} \Rightarrow s = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{2\,j\sin\omega/2}{2\cos\omega/2} = j\tan\frac{\omega}{2}$$

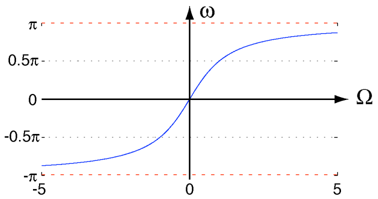
u.circle

im.axis

- i.e.

$$\Omega = \tan\left(\tfrac{\omega}{2}\right)$$

$$\omega = 2\tan^{-1}\Omega$$

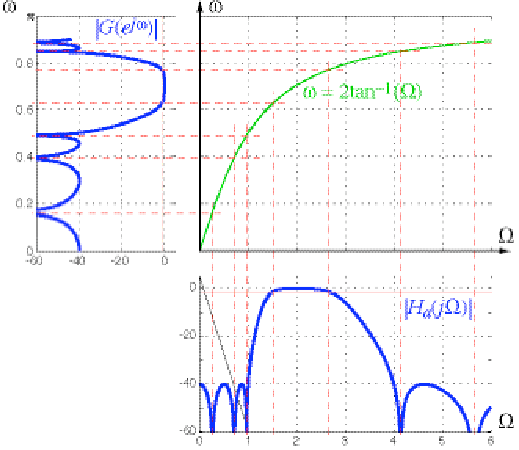


- *infinite* range of CT frequency $-\infty < \Omega < \infty$ maps to *finite* DT freq. range $-\pi < \omega < \pi$
- nonlinear; $\frac{d}{d\omega}\Omega \to \infty$ as ω → π

pack it all in!

Frequency Warping

- Bilinear transform makes
  $G\left(e^{j\omega}\right) = H_a(j\Omega)\big|_{\omega = 2\tan^{-1}\Omega}$ for all ω, Ω



- Same gain & phase (ε, A...), in same ‘order’, but with *warped* frequency axis

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Design Procedure

- Obtain DT filter specs:
  - general form (LP, HP...), $\omega_p, \omega_s, \frac{1}{\sqrt{1+\varepsilon^2}}, \frac{1}{A}$
- ‘Warp’ *frequencies* to CT:
  - $\Omega_p = \tan\frac{\omega_p}{2} \quad \Omega_s = \tan\frac{\omega_s}{2}$

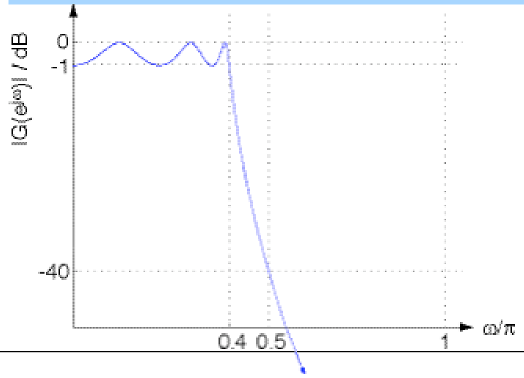
Old-style

- Design analog filter for $\Omega_p, \Omega_s, \frac{1}{\sqrt{1+\varepsilon^2}}, \frac{1}{A}$
  - $\rightarrow H_a(s)$, CT filter polynomial
- Convert to DT domain: $G(z) = H_a(s)\big|_{s = \frac{1-z^{-1}}{1+z^{-1}}}$
  - $\rightarrow G(z)$, rational polynomial in z
- Implement digital filter!

Bilinear Transform Example

- DT domain requirements:

  Lowpass, 1 dB ripple in PB, $\omega_p = 0.4\pi$,
  SB attenuation ≥ 40 dB @ $\omega_s = 0.5\pi$,
  attenuation increases with frequency



- PB ripples,
  SB monotonic
  → Chebyshev I

Bilinear Transform Example

- Warp to CT domain:

$$\Omega_p = \tan\frac{\omega_p}{2} = \tan 0.2\pi = 0.7265 \text{ rad/sec}$$

$$\Omega_s = \tan\frac{\omega_s}{2} = \tan 0.25\pi = 1.0 \quad \text{rad/sec}$$

- Magnitude specs:

  1 dB PB ripple

$$\Rightarrow \frac{1}{\sqrt{1+\varepsilon^2}} = 10^{-1/20} = 0.8913 \Rightarrow \varepsilon = 0.5087$$

  40 dB SB atten.

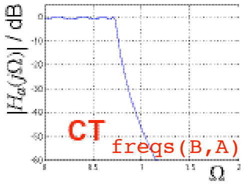
$$\Rightarrow \frac{1}{A} = 10^{-40/20} = 0.01 \Rightarrow A = 100$$

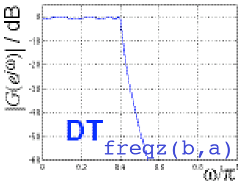
Bilinear Transform Example

- Chebyshev I design criteria:

$$N \geq \frac{\cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = 7.09 \qquad \text{i.e. need } N = 8$$

- Design analog filter, map to DT, check:

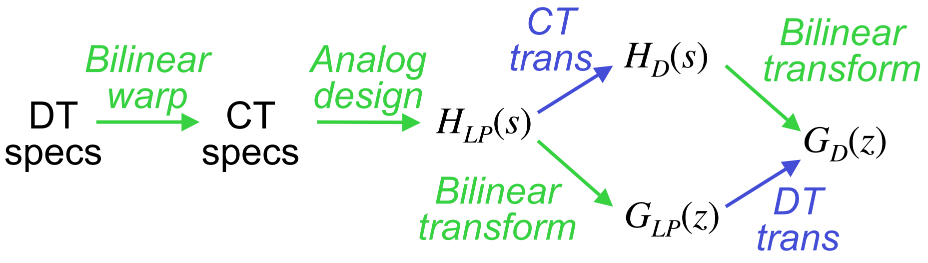




```
>> N=8;
>> wp=0.7265;
>> [B,A]=cheby1(N,1,wp,'s');
>> [b,a] = bilinear(B,A,.5);
```

Other Filter Shapes

- Example was IIR LPF from LP prototype
- For other shapes (HPF, bandpass,...):



$$\begin{tikzcd}
 & & & H_D(s) \arrow[rd, "\text{Bilinear transform}"] & \\
\text{DT specs} \arrow[r, "\text{Bilinear warp}"] & \text{CT specs} \arrow[r, "\text{Analog design}"] & H_{LP}(s) \arrow[ru, "\text{CT trans}"] \arrow[rd, "\text{Bilinear transform}"'] & & G_D(z) \\
 & & & G_{LP}(z) \arrow[ru, "\text{DT trans}"'] &
\end{tikzcd}$$

- Transform LP→X in CT *or* DT domain...

DT Spectral Transformations

- Same idea as CT LPF→HPF mapping, but in z-domain:

$$G_D(\hat{z}) = G_L(z)\big|_{z=F(\hat{z})} = G_L(F(\hat{z}))$$

- To behave well, $z = F(\hat{z})$ should:
  - map u.c. → u.c. (preserve $G(e^{j\omega})$ values)
  - map u.c. interior → u.c. interior (stability)
- i.e. $|F(\hat{z})| = 1 \leftrightarrow |\hat{z}| = 1 \qquad |F(\hat{z})| < 1 \leftrightarrow |\hat{z}| < 1$
  - in fact, $F(\hat{z})$ matches the definition of an allpass filter ... replace delays with $F(\hat{z})^{-1}$

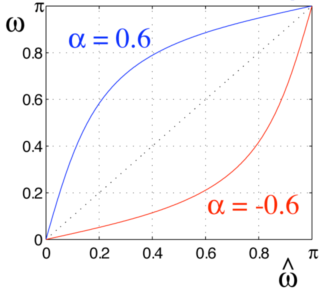
DT Frequency Warping

- Simplest mapping $z = F(\hat{z}) = \frac{\hat{z}-\alpha}{1-\alpha\hat{z}}$ has effect of warping frequency axis:

$$\hat{z} = e^{j\hat{\omega}} \Rightarrow z = e^{j\omega} = \frac{e^{j\hat{\omega}} - \alpha}{1 - ae^{j\hat{\omega}}}$$

α > 0 : expand HF

$$\Rightarrow \tan\left(\frac{\omega}{2}\right) = \frac{1+\alpha}{1-\alpha}\tan\left(\frac{\hat{\omega}}{2}\right)$$



α < 0 : expand LF

Another Design Example

- Spec:
  - Bandpass, from 800-1600 Hz (SR = 8kHz)
  - Ripple = 1dB, min. stopband atten. = 60 dB
  - 8th order, best transition band
- Use elliptical for best performance
- Full design path:
  - design analog LPF prototype
  - analog LPF → BPF
  - CT BPF → DT BPF (Bilinear)

Another Design Example

- Or, do it all in one step in Matlab:

```
[b,a] = ellip(8,1,60,
              [800 1600]/(8000/2));
```